Bessel functions of integer order in terms of hyperbolic functions

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Abstract

Using Jacobi's identity, a simple formula expressing Bessel functions of integer order as simple combinations of powers and hyperbolic functions, plus higher order corrections, is obtained.

In this article we shall propose a simple formula expressing the modified Bessel functions of integer order, I_n, in terms of powers and hyperbolic functions of the same argument. It can be easily adapted for the Bessel functions J_n.

The starting point is the generalization of the Jacobi identity ([1] p.22) used in the calculation of lattice sums [2]:

$$\frac{1}{N}\sum_{m=0}^{N-1}\exp\left[\frac{z}{2}\left(we^{i\frac{2\pi m}{N}}+\frac{1}{w}e^{-i\frac{2\pi m}{N}}\right)\right]=\sum_{k=-\infty}^{\infty}w^{kN}\cdot I_{kN}(z). \tag{1}$$

For $w=1$ and $N=2$, (1) gives a well-known formula:

$$\cosh z=\sum_{k=-\infty}^{\infty}I_{2k}(z)=I_0(z)+2\left[I_2(z)+I_4(z)+...\right], \tag{2}$$

(see for instance [3], eq.9.6.39)

It is interesting to exploit (1) at $w=1$, for larger values of N. For $N=4$,

$$\frac{1}{2}\left(1+\cosh z\right)=\cosh^2\frac{z}{2}=I_0(z)+2\sum_{k=1}^{\infty}I_{4k}(z), \tag{3}$$

and for $N=8$:

$$\frac{1}{4}(1+\cosh z+2\cosh\frac{z}{\sqrt{2}})=I_0(z)+2\sum_{k=1}^{\infty}I_{8k}(z). \tag{4}$$

It is easy to see that, if the l.h.s. of these equations contains p hyperbolic cosines, it provides an exact expression for the series of $I_0(z)$, cut off at the z^{4p} term. The generalization of (4) for $N=4p$, with p - an arbitrary integer, is indeed:

$$\frac{1}{2p}\left[1+\cosh z+2\cosh\left(z\cos\frac{\pi}{2p}\right)+...+2\cosh\left(z\cos\left(p-1\right)\frac{\pi}{2p}\right)\right]=I_0(z)+2\sum_{k=1}^{\infty}I_{4pk}(z). \tag{5}$$

Because

$$I_n(-iz)=i^{-n}J_n(z). \tag{6}$$

our result (5) can be written as:

$$\frac{1}{2p}\left[1+\cos z+2\cos\left(z\cos\frac{\pi}{2p}\right)+...+2\cos\left(z\cos\left(p-1\right)\frac{\pi}{2p}\right)\right]= \tag{7}$$

$$= J_0(z) + 2\sum_{k=1}^{\infty} J_{4pk}(z).$$

It is easy to obtain formulae similar to (5) for any modified Bessel function of integer order. Let us introduce the notations:

$$c_1 = \cos\frac{\pi}{2p}, \qquad ... \qquad c_{p-1} = \cos\frac{p-1}{2p}\pi. \tag{8}$$

and let us define the functions:

$$S_q\left(z\right) = \sinh z + 2\left(c_1\right)^q \sinh\left(c_1 z\right) + ... + 2\left(c_p\right)^q \sinh\left(c_p z\right), \qquad q > 0 \tag{9}$$

$$C_q\left(z\right) = \cosh z + 2\left(c_1\right)^q \cosh\left(c_1 z\right) + ... + 2\left(c_p\right)^q \cosh\left(c_p z\right), \qquad q \geq 0 \tag{10}$$

We have:

$$\frac{dS_q\left(z\right)}{dz} = C_{q+1}\left(z\right); \qquad \frac{dC_q\left(z\right)}{dz} = S_{q+1}\left(z\right) \tag{11}$$

Using recurrently the formula ([4] 8.486.4):

$$\frac{d}{dz}I_n\left(z\right) - \frac{n}{z}I_n\left(z\right) = I_{n+1}\left(z\right) \tag{12}$$

and the notation:

$$T_n\left(z\right) = \left(\frac{1}{z}\frac{d}{dz}\right)^n C_0\left(z\right), \tag{13}$$

we get:

$$\frac{1}{2p}z^n T_n = I_n\left(z\right) + 2z^n\left(\frac{1}{z}\frac{d}{dz}\right)^n \sum_{k=1}^{\infty} I_{4pk}(z). \tag{14}$$

For $n = 1, 2, 3, 4$:

$$T_1\left(z\right) = z^{-1}S_1\left(z\right), \; T_2 = -z^{-3}S_1\left(z\right) + z^{-2}C_2\left(z\right) \;, \tag{15}$$

$$T_3\left(z\right) = 3z^{-5}S_1\left(z\right) - 3z^{-4}C_2\left(z\right) + z^{-3}S_3\left(z\right), \tag{16}$$

$$T_4(z) = -15z^{-7}S_1(z) + 15z^{-6}C_2(z) - 6z^{-5}S_3(z) + z^{-4}C_4(z). \tag{17}$$

The general expressions are:

$$T_{2n} = z^{-2n}\left[\alpha_1^{(2n)}z^{-2n+1}S_1 + \alpha_2^{(2n)}z^{-2n+2}C_2 + ... + \alpha_{2n}^{(2n)}C_{2n}\right], \tag{18}$$

$$T_{2n+1} = z^{-2n-1}\left[\alpha_1^{(2n+1)}z^{-2n}S_1 + \alpha_2^{(2n+1)}z^{-2n+1}C_2 + ... + \alpha_{2n+1}^{(2n+1)}S_{2n+1}\right], \tag{19}$$

We get:

$$\alpha_1^{(n)} = -\alpha_2^{(n)} = (-1)^{n+1}(2n-3)!!, \qquad \alpha_{n-1}^{(n)} = -\frac{(n-1)\,n}{2}, \qquad \alpha_n^{(n)} = 1. \tag{20}$$

and the following recurrence relations for the coefficients $\alpha_q^{(p)}$:

$$\alpha_{n-p}^{(n)} = \alpha_{n-p-1}^{(n-1)} - (n+p-2)\,\alpha_{n-p}^{(n-1)}, \qquad 2 \leqslant p \leqslant n-3. \tag{21}$$

Other general expressions of the coefficients are:

$$\alpha_3^{(n)} = (-1)^{n+1}(n-2)(2n-5)!! \tag{22}$$

$$\alpha_4^{(n)} = (-1)^n (n-3)!\left\{\frac{2^{n-4}}{0!}1!! + \frac{2^{n-3}}{1!}3!! + ... + \frac{2^0}{(n-4)!}(2n-7)!!\right\}. \tag{23}$$

Ignoring the series in the r.h.s. of (14), we get approximate expression for I_n. Let us give here these expressions for the value $p = 2$ and $n = 0, 1, 2, 3$:

$$I_0^{(ap)}(z) = \frac{1}{4}\cdot\left(1 + \cosh z + 2\cosh\frac{z}{\sqrt{2}}\right), \tag{24}$$

$$I_1^{(ap)}(z) = \frac{1}{4}\cdot\left(\sinh z + \sqrt{2}\sinh\frac{z}{\sqrt{2}}\right), \tag{25}$$

$$I_2^{(ap)}(z) = \frac{1}{4}\cdot\left(-\frac{1}{z}\left(\sinh z + \sqrt{2}\sinh\frac{z}{\sqrt{2}}\right) + \cosh z + \cosh\frac{z}{\sqrt{2}}\right), \tag{26}$$

$$I_3^{(ap)}(z) = \frac{1}{4}\left[\frac{3}{z^2}\left(\sinh z + \sqrt{2}\sinh\frac{z}{\sqrt{2}}\right) - \frac{3}{z}\left(\cosh z + \cosh\frac{z}{\sqrt{2}}\right)\right.$$

$$-\frac{3}{z}\left(\cosh z+\cosh\frac{z}{\sqrt{2}}\right)+\left(\sinh z+\frac{1}{\sqrt{2}}\sinh\frac{z}{\sqrt{2}}\right)\Bigg].\tag{27}$$

According to the Table 1, even for this very small value of p, the approximation provided by these functions, for "moderate" values of the argument ($z \lesssim 4$), is very good.

Table 1

z	1	2	3	4
$\frac{I_0^{(ap)}-I_0}{I_0}$	1.6×10^{-7}	2.4×10^{-5}	3.3×10^{-4}	1.7×10^{-3}
$\frac{I_1^{(ap)}-I_1}{I_1}$	2.3×10^{-6}	1.4×10^{-4}	1.2×10^{-3}	4.4×10^{-3}
$\frac{I_2^{(ap)}-I_2}{I_2}$	7.1×10^{-5}	10^{-3}	4.5×10^{-3}	1.2×10^{-2}
$\frac{I_3^{(ap)}-I_3}{I_3}$	1.8×10^{-3}	7.3×10^{-3}	1.7×10^{-2}	3×10^{-2}

It is visible that the precision of the approximation decreases with the order n of the Bessel function I_n. We can increase it arbitrarly, by increasing the value of p.

The extension of these results to Bessel functions of real argument is trivial, using the formula (6) and:

$$S_q(-iz)=-i\left[\sin z+2c_1^q\sin(c_1z)+...+2c_p^q\sin(c_pz)\right],\tag{28}$$

$$C_q(-iz)=\cos z+2c_1^q\cos(c_1z)+...+2c_p^q\cos(c_pz).\tag{29}$$

In conclusion, we have proposed a controlled analytic approximation for Bessel functions of integer order. The first $4p-n$ terms of the series representation of I_n is generated exactly by the first $4p-n$ terms of the elementary functions in the l.h.s. of eq.(14). So, our formulae can be used, for instance, to find the series expansions of the powers of Bessel functions, a subject discussed recently by Bender et al [5]

The results presented in this paper can be applied to a large variety of problems, mainly with cylindrical symmetry, involving Bessel functions at "moderate" arguments. They may provide also a useful "visualization" of J_n and I_n in terms of elementary functions. The method cannot be used for asymptotic problems.

[1] G.N.Watson 1948 A Treatise on the Theory of Bessel Functions (Cambridge University Press)

[2] Cojocaru S 2006 *Int. J. Mod. Phys.* **20** 593

[3] Abramowitz M and Stegun I A 1972 Handbook of Mathematical Functions (New York: Dover Publications)

[4] Gradstein I S and Ryzhik I M 1980 Tables of Integrals, Series and Products (New York, London: Academic Press)

[5] Bender C M, Brody D C and Meister B K 2003 *J.Math.Phys.* **44** 309